

Einstein's Equations and Equivalent Hyperbolic Dynamical Systems *

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Abstract

We discuss several explicitly causal hyperbolic formulations of Einstein's dynamical $3 + 1$ equations in a coherent way, emphasizing throughout the fundamental role of the “slicing function,” α —the quantity that relates the lapse N to the determinant of the spatial metric \bar{g} through $N = \bar{g}^{1/2}\alpha$. The slicing function allows us to demonstrate explicitly that every foliation of spacetime by spatial time-slices can be used in conjunction with the causal hyperbolic forms of the dynamical Einstein equations. Specifically, the slicing function plays an essential role (1) in a clearer form of the canonical action principle and Hamiltonian dynamics for gravity and leads to a recasting (2) of the Bianchi identities $\nabla_\beta G^\beta_\alpha \equiv 0$ as a well-posed system for the evolution of the gravitational constraints in vacuum, and also (3) of $\nabla_\beta T^\beta_\alpha \equiv 0$ as a well-posed system for evolution of the energy and momentum components of the stress tensor in the presence of matter, (4) in an explicit rendering of four hyperbolic formulations of Einstein's equations with only physical characteristics, and (5) in providing guidance to a new “conformal thin sandwich” form of the initial value constraints.

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1 Introduction

Einstein's equations have, for much of the history of general relativity, been explored very fruitfully in terms of their concise and elegant statements characterizing the geometry of four dimensional pseudo-riemannian geometries. Such geometries depict possible physical spacetimes containing only "the gravitational field itself." The variety and properties of these "empty" spacetimes is truly astonishing. Quasi-local geometric entities such as trapped surfaces and event horizons have become familiar. It is now firmly established that large-scale topological and geometrical features of spacetime are, indeed, subjects of physical inquiry. The nature and distribution of "matter" at stellar scales and upward has also brought particle physics and hydrodynamics to the fore.

During these years, however, a steady development of the "space-plus-time" or $3 + 1$ view of spacetime geometry has also occurred. Here one views general relativity as "geometrodynamics" in the parlance of John Wheeler [1]. The emphasis, in the canonical or Hamiltonian explication of geometrodynamics given by Arnowitt, Deser, and Misner ("ADM") [2] and by Dirac [3, 4], is on the evolving intrinsic and extrinsic geometry of spacelike hypersurfaces which determine, by knowledge of the appropriate initial data and by classical causality, the spacetime "ahead" (and "behind"), if spacetime is globally hyperbolic, an assumption we adopt throughout.

Underlying and preceding geometrodynamics and Hamiltonian methods, however, was the basic realization that four of the ten Einstein vacuum equations are nonlinear constraints on the initial Cauchy data, which play such a decisive role in defining the later canonical formalism [5]. The Cauchy problem, constraints plus evolution, was shown to be well-posed in the modern sense of nonlinear partial differential equations [6, 7, 8, 9]. This train of progress was marked by early work of Darmois and Lichnerowicz [8], and brought to definitive development by one of us [6, 7].

At this writing, with accurate three-dimensional simulations using the full Einstein equations, with and without the presence of stress-energy sources, becoming essential for realistic studies of gravity waves, high energy astrophysics, and early cosmology, studies of Einstein's equations of evolution in $3 + 1$ form have blossomed [11-28] (an incomplete sample — see also [10]). Hyperbolic forms, especially first-order symmetrizable forms possessing only physically causal directions of propagation, have undergone very significant development in the past few years.

Specifically, in this chapter we describe in detail several explicitly causal hyperbolic formulations of Einstein's dynamical $3 + 1$ equations by following a path that can be viewed as lighted by the “slicing function,” α — the quantity that relates the lapse N to the determinant of the spatial metric \bar{g} through $N = \bar{g}^{1/2}\alpha$. This representation of the lapse function was presented in [18]. The slicing function allows us to demonstrate explicitly that no foliation of spacetime by spatial time-slices can be an obstacle to the causal hyperbolic forms of the dynamical Einstein equations. The slicing function plays an essential role (1) in a more precise form of the canonical action principle and canonical dynamics for gravity, (2) leads to a recasting of the Bianchi identities $\nabla_\beta G^\beta_\alpha \equiv 0$ as a well-posed system for the evolution of the gravitational constraints in vacuum and also (3) of $\nabla_\beta T^\beta_\alpha \equiv 0$ as a well-posed system for evolution of the energy and momentum components of the stress tensor in the presence of matter, (4) in an explicit display of four hyperbolic formulations of Einstein's equations with only physical characteristics, and (5) even in providing guidance to a new elliptic “conformal thin sandwich” form of the initial value constraints.

We recall that the proof of the existence of a causal evolution in local Sobolev spaces of \bar{g} and its extrinsic curvature (second fundamental tensor) K into an Einsteinian spacetime does not result directly from the equations giving the time derivatives of \bar{g} and K in terms of space derivatives of these quantities in a straightforward $3 + 1$ decomposition of the Ricci tensor of the spacetime metric, which contains also the lapse and shift characterizing the time lines. These equations do not appear as a hyperbolic system for arbitrary lapse and shift, in spite of the fact that their characteristics are only the light cone and the time axis [25].

We now turn to notational matters, conventions, and to the $3 + 1$ decomposition of the Riemann and Ricci tensors. We assume here and throughout the sequel that the spacetime $V = M \times \mathbb{R}$ is endowed with a metric g of signature $(-, +, +, +)$ and that the time slices are spacelike, that is, have signature $(+, +, +)$. These assumptions are not restrictive for globally hyperbolic (pseudo-riemannian) spacetimes.

We choose on V a moving coframe such that the dual vector frame has a time axis orthogonal to the slices M_t while the space axes are tangent to them. Specifically, we set

$$\begin{aligned}\theta^0 &= dt, \\ \theta^i &= dx^i + \beta^i dt,\end{aligned}\tag{1}$$

with $t \in \mathbb{R}$ and x^i , $i = 1, 2, 3$ local coordinates on M . The Pfaff or convective derivatives ∂_α with respect to θ^α are

$$\begin{aligned}\partial_0 &\equiv \frac{\partial}{\partial t} - \beta^i \partial_i \\ \partial_i &\equiv \frac{\partial}{\partial x^i}\end{aligned}\tag{2}$$

In this coframe, the metric g reads

$$ds^2 = g_{\alpha\beta} \theta^\alpha \theta^\beta \equiv -N^2 (\theta^0)^2 + g_{ij} \theta^i \theta^j .\tag{3}$$

The t -dependent scalar N and space vector β are called the lapse function and shift vector of the slicing. These quantities were explicitly identified in [7] and play prominent roles in all subsequent $3 + 1$ formulations. Any spacetime tensor decomposes into sets of time dependent space tensors by projections on the tangent space or the normal to M_t .

We define for any t -dependent space tensor T another such tensor of the same type, $\bar{\partial}_0 T$, by setting

$$\bar{\partial}_0 \equiv \frac{\partial}{\partial t} - \mathcal{L}_\beta ,\tag{4}$$

where \mathcal{L}_β is the Lie derivative on M_t with respect to β .

Notice that in our foliation-adapted basis (1) and (2), that if g denotes the spacetime metric and \bar{g} the space metric, then we have ($g_{0i} = g^{0i} = 0$ in our frames):

$$g_{ij} = \bar{g}_{ij} ; \quad g^{ij} = \bar{g}^{ij} .\tag{5}$$

(Greek indices range $\{0, 1, 2, 3\}$ while latin ones are purely spatial.) Hence, *no overbars will be used to denote components of the spatial metric*. On the other hand, for the determinants, we have $(-\det g) = N^2(\det \bar{g})$. Therefore we shall use overbars on spatial metric determinants; for example $\bar{g}^{1/2} \equiv (\det \bar{g})^{1/2}$.

Likewise, to distinguish the purely spatial components of the spacetime Ricci tensor (say), we shall write $R_{ij}(g)$, while for the space Ricci tensor we shall write $R_{ij}(\bar{g})$. In general, of course, $R_{ij}(g) \neq R_{ij}(\bar{g})$. The Levi-Civita connection of g is denoted by ∇ and that of \bar{g} by $\bar{\nabla}$.

With the convention

$$\nabla_\alpha \sigma_\beta \equiv \partial_\alpha \sigma_\beta - \sigma_\rho \gamma^\rho{}_{\beta\alpha} ,\tag{6}$$

and the definitions

$$\gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + g^{\alpha\delta} C^\varepsilon_{\delta(\beta} g_{\gamma)\varepsilon} - \frac{1}{2} C^\alpha{}_{\beta\gamma} , \quad (7)$$

$$d\theta^\alpha = -\frac{1}{2} C^\alpha{}_{\beta\gamma} \theta^\beta \wedge \theta^\gamma , \quad (8)$$

we have for the connection coefficients (Γ denotes an ordinary Christoffel symbol)

$$\gamma^i{}_{jk} = \Gamma^i{}_{jk}(g) = \Gamma^i{}_{jk}(\bar{g}) \quad (9)$$

$$\gamma^i{}_{0k} = -N K^i{}_k , \quad \gamma^i{}_{j0} = -N K^i{}_k + \partial_j \beta^i , \quad \gamma^0{}_{ij} = -N^{-1} K_{ij} \quad (10)$$

$$\gamma^i{}_{00} = N \partial^i N , \quad \gamma^0{}_{0i} = \gamma^0{}_{i0} = \partial_i \log N , \quad \gamma^0{}_{00} = \partial_0 \log N . \quad (11)$$

Observe that if α is a space scalar of weight -1 , we have

$$\bar{\nabla}_i \alpha = \partial_i \alpha + \alpha \Gamma^k{}_{ki}(g) = \partial_i \alpha + \alpha \partial_i \log \bar{g}^{1/2} , \quad (12)$$

$$\mathcal{L}_\beta \alpha = \beta^i \bar{\nabla}_i \alpha + \alpha \bar{\nabla}_i \beta^i . \quad (13)$$

The Riemann tensor is fixed by

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\gamma = V^\delta R^\gamma{}_{\delta\alpha\beta} \quad (14)$$

while the Ricci tensor is $R_{\delta\beta} \equiv R^\gamma{}_{\delta\gamma\beta}$.

The 3 + 1 decompositions of the Riemann and Ricci tensors are

$$R_{ijkl}(g) = R_{ijkl}(\bar{g}) + 2K_{i[k} K_{l]j} , \quad (15)$$

$$R_{0ijk}(g) = 2N \bar{\nabla}_{[j} K_{k]i} , \quad (16)$$

$$R_{0i0j}(g) = N (\bar{\partial}_0 K_{ij} + N K_{ik} K^k{}_j + \bar{\nabla}_i \partial_j N) . \quad (17)$$

One can then obtain for the Ricci tensor

$$R_{ij}(g) = R_{ij}(\bar{g}) - N^{-1} \bar{\partial}_0 K_{ij} + K K_{ij} - 2K_{ik} K^k{}_j - N^{-1} \bar{\nabla}_i \partial_j N , \quad (18)$$

$$R_{0j}(g) = N (\partial_j K - \bar{\nabla}_h K^h{}_j) , \quad (19)$$

$$R_{00}(g) = N (\partial_0 K - N K_{ij} K^{ij} + \Delta_{\bar{g}} N) , \quad (20)$$

where $K \equiv K^i{}_i$ and $\Delta_{\bar{g}} \equiv g^{ij} \bar{\nabla}_i \bar{\nabla}_j$. Finally, we note

$$G^0{}_0 = \frac{1}{2} (K_{ij} K^{ij} - K^2 - R(\bar{g})) . \quad (21)$$

2 Every Time Slicing Is “Harmonic”

The standard statement of the harmonic time-slicing condition is, that on a $t = \text{const.}$ time slice, $\bar{\partial}_0[(-g)^{1/2}g^{00}] = 0$. (This is equivalent, in a coordinate basis, to $\partial_\mu[(-g)^{1/2}g^{\mu t}] = 0$.) Friedrich observed in [24] that the right hand side of these equations could be a given function of (t, x^i) ¹. (See also [19].) Therefore, the standard harmonic condition expressed in 3 + 1 form, $\bar{\partial}_0 N + N^2 K = 0$, can be written as a generalized “harmonic” condition

$$\bar{\partial}_0 N + N^2 K = N f , \quad (22)$$

where $f(t, x)$ is a known function. Specifically, introduce $\alpha(x, t)$ such that $\bar{\partial}_0 \log \alpha = f$, then (22) becomes

$$\bar{\partial}_0 N + N^2 K = N \bar{\partial}_0 \log \alpha , \quad (23)$$

from which the identity

$$\bar{\partial}_0 \log \bar{g}^{1/2} = -NK , \quad (24)$$

allows us to see that

$$N = \bar{g}^{1/2} \alpha . \quad (25)$$

We shall call $\alpha(x, t)$ the “slicing function;” it is a freely given scalar density of weight -1 .

It is clear that any $N > 0$ on a given time slice $t = t_0$ can be written in the form $N_{t_0} = \bar{g}_{t_0}^{1/2} \alpha(t_0, x)$ for some $\alpha > 0$ provided that $g_{ij}(t_0, x)$ is a proper riemannian metric. Introducing “harmonic” time-slicing is thus a simple matter. It is not, however, known at present how to construct a specific long-time foliation from general rules telling how to specify $\alpha(t, x)$. However, many foliations can be constructed in a “step-by step” fashion (numerical time steps) provided certain obvious conditions are met. For example, an elliptic condition on N can determine $\alpha(t)$ on a sequence of time slices if the condition does not couple to variables that disturb the characteristic directions of the hyperbolic equations. (The same is true for the shift vector β^i .) Alternatively, we can try educated guesses for $\alpha(t, x)$.

As it stands, (23) is clearly a speed zero (with respect to ∂_0) hyperbolic equation. However, this equation and Einstein’s equations lead to a second

¹Just as in electrodynamics, $\nabla^\mu A_\mu = 0 \rightarrow \nabla^\mu A_\mu{}' = \ell(t, x) \neq 0$ is perfectly acceptable as a “Lorentz gauge” if ℓ is known.

order equation in space and time that propagates N along the light cone. This result brings into sharp relief the congruence of (23) with the propagation on the light cone of other variables.

The trace of $R_{ij}(g)$ gives an equation for $\bar{\partial}_0 K$ [19]

$$\bar{\partial}_0 K = -\Delta_{\bar{g}} N + [R(\bar{g}) + K^2 - R^k_k(g)] N, \quad (26)$$

where $\Delta_{\bar{g}}$ in (26) denotes the Laplacian $g^{ij}\bar{\nabla}_i\bar{\nabla}_j$. Taking the time derivative of (23) and eliminating $\bar{\partial}_0 K$ with (26) shows that N obeys the non-linear wave equation

$$\bar{\square}_g N + R^k_k(g)N - R(\bar{g})N - N\bar{\partial}_0 \log \alpha + (\bar{\partial}_0^2 \log \alpha)N^{-1} = 0, \quad (27)$$

where we wrote our wave operator or “d’Alembertian” as $\bar{\square}_g = -(N^{-1}\bar{\partial}_0)^2 + \Delta_{\bar{g}}$. The characteristic cone of $\bar{\square}_g$ is clearly the physical light cone ($c = 1$). The equation (27) *per se* will not be used explicitly in the sequel.

Substitutions of the form $N_\lambda = \bar{g}^{\lambda/2}\alpha_\lambda$ ($\lambda > 0$) have also been considered [27]. However, after working out the wave equation analogous to (27) that N_λ obeys, one finds that the local proper propagation speed of N_λ is $\sqrt{\lambda}$. This behavior may or may not spoil the propagation of system variables other than N , but if $\lambda \neq 1$ and the system is hyperbolic, one will always find that the characteristic directions of the system will not all be physical ones. That is, in vacuum gravity, there will be some variables that propagate neither on the light cone (speed = 1) nor along the axis parallel to $\bar{\partial}_0$ (speed = 0) which is orthogonal to $t = \text{const.}$ The variables not propagating in physical directions are gauge variables and one will not have physical criteria for their boundary values on characteristic surfaces. On the other hand, with $\lambda = 1$, one has fulfilled a necessary condition that physical and gauge variables propagate together in the same directions.

In the following sections, whenever we consider hyperbolic systems, we will focus on first-order symmetric (or symmetrizable) hyperbolic (“FOSH”) equations possessing *only* physical characteristic directions. We understand “FOSH” in *this restricted physical sense only* in this paper, and likewise for other uses of the term “hyperbolic.”

3 Canonical Action and Equations of Motion

Choice of the slicing function α in (25) is arbitrary ($\alpha > 0$). That α is freely chosen while N must satisfy an equation of motion (23) suggests that

it should be regarded as the undetermined² multiplier in the canonical action principle of Arnowitt, Deser, and Misner (“ADM”) [2]. Here we follow [29].

To draw some lessons for the canonical formalism, let us first express the $3 + 1$ evolution equations in their standard geometrical form (see: with zero shift [8], arbitrary lapse and shift [7], spacetime perspective [30]):

$$\dot{g}_{ij} \equiv -2NK_{ij} , \quad (28)$$

$$\dot{K}_{ij} \equiv N \left(-R_{ij}(g) + R_{ij}(\bar{g}) + NK_{ij} - K_{ik}K_j^k - N^{-1}\bar{\nabla}_i\partial_j N \right) , \quad (29)$$

where $(\dot{}) \equiv \bar{\partial}_0()$.

A brief look at (29) shows that forming the combination $\mathcal{R}_{ij} = R_{ij}(g) - g_{ij}R^k_k(g)$ leads to an equation of motion for the ADM canonical momentum

$$\pi^{ij} = \bar{g}^{1/2} (Kg^{ij} - K^{ij}) \quad (30)$$

that contains no constraints. (In this section we choose units in which $16\pi G = c = 1$.) Indeed, using (28) and (29), we obtain the identity

$$\begin{aligned} \dot{\pi}^{ij} \equiv & N\bar{g}^{1/2} (R(\bar{g})g^{ij} - R^{ij}(\bar{g})) - N\bar{g}^{-1/2} (2\pi^{ik}\pi_k^j - \pi\pi^{ij}) \\ & + \bar{g}^{1/2} (\bar{\nabla}^i\bar{\nabla}^j N - g^{ij}\bar{\nabla}_k\bar{\nabla}^k N) + N\bar{g}^{1/2} [\mathcal{R}^{ij}] . \end{aligned} \quad (31)$$

From the identity $\dot{g}_{ij} = -2NK_{ij}$, we have

$$\dot{g}_{ij} \equiv N\bar{g}^{-1/2} (2\pi_{ij} - \pi g_{ij}) . \quad (32)$$

We now come to a crucial observation. Were the canonical equation for $\dot{\pi}^{ij}$ to be dictated by vanishing of the spatial part of the Einstein tensor, $G^{ij}(g) = 0$, as it is in the conventional ADM analysis [2], then the identity

$$G_{ij}(g) + g_{ij}G^0_0(g) \equiv R_{ij}(g) - g_{ij}R^k_k(g) \equiv \mathcal{R}_{ij} \quad (33)$$

shows that a Hamiltonian constraint term $\sim \bar{g}^{1/2}G^0_0$ remains in the $\dot{\pi}^{ij}$ equation (31). This would mean that the validity of the $\dot{\pi}^{ij}$ equation would be restricted to the subspace on which the Hamiltonian constraint is satisfied (i.e., vanishes).

²The multipliers associated with gauge freedom or, as here, with spacetime coordinate freedom, can be freely chosen because they are not determined by physical conditions. Hence, they are not true “Lagrange multipliers.”

Though the ADM derivation of the $\dot{\pi}^{ij}$ equation, found by varying g_{ij} in their canonical action (β^i is the shift vector)

$$S[g, \pi; N, \beta] = \int d^4x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H}) , \quad (34)$$

with $N(t, x)$, $\beta^i(t, x)$ and π^{ij} held fixed, is of course perfectly correct, another point of view is possible. [We are ignoring boundary terms, a subject not of interest here, and we note that the momentum constraint term $-\beta^i \mathcal{H}_i$ ($\mathcal{H}_i = g^{1/2} \mathcal{C}_i$, $\mathcal{C}_i = 2NR^0_i$) is contained in $\pi^{ij} \dot{g}_{ij}$ ($(\dot{}) \equiv \bar{\partial}_0$) upon integration by parts.] (The slicing density α has also been used prominently in the action by Teitelboim [31], who simply set $\alpha = 1$ ($N = \bar{g}^{1/2}$), and by Ashtekar [32, 33] for other purposes.)

We have explained that α can be regarded as a free undetermined multiplier while N is a *dynamical variable* (a conclusion also reached by Ashtekar for other reasons [32, 33]) that determines the proper time $N\delta t$ between slices $t = t'$ and $t = t' + \delta t$. N is determined from $\alpha(t, x)$ and $\bar{g}^{1/2}$ found by solving the initial value constraint equations. (See the treatment of the constraints in the final section of this article and in [30, 34, 35, 9].) Motivated by this viewpoint, we alter the undetermined multiplier N in the ADM action principle to α , where the Hamiltonian density \mathcal{H} is (with $\mathcal{H} \equiv 2\bar{g}^{1/2}G^0_0(g)$ being the ADM Hamiltonian density of weight +1)

$$\tilde{\mathcal{H}} \equiv \bar{g}^{1/2} \mathcal{H} = \pi_{ij} \pi^{ij} - \frac{1}{2} \pi^2 - \bar{g} R(\bar{g}) , \quad (35)$$

a scalar density of weight +2 and a rational function of the metric. The action becomes

$$S[\bar{g}, \pi; \alpha, \beta] = \int d^4x (\pi^{ij} \dot{g}_{ij} - \alpha \tilde{\mathcal{H}}) . \quad (36)$$

The modified action *principle* for the canonical equations that we propose in (36) is to vary π^{ij} and g_{ij} , with $\alpha(t, x)$ and $\beta^i(t, x)$ as fixed undetermined multipliers. From

$$\begin{aligned} \delta \tilde{\mathcal{H}} &= (2\pi_{ij} - g_{ij}\pi) \delta \pi^{ij} + (2\pi^{ik} \pi^j_k - \pi \pi^{ij} + \bar{g} R^{ij}(\bar{g}) - \bar{g} g^{ij} R(\bar{g})) \delta g_{ij} \\ &\quad - \bar{g} (\bar{\nabla}^i \bar{\nabla}^j \delta g_{ij} - g^{ij} \bar{\nabla}_k \bar{\nabla}^k \delta g_{ij}) , \end{aligned} \quad (37)$$

we obtain the canonical equations

$$\dot{g}_{ij} = \alpha \frac{\delta \tilde{\mathcal{H}}}{\delta \pi^{ij}} = \alpha (2\pi_{ij} - \pi g_{ij}) \equiv -2N K_{ij} , \quad (38)$$

$$\begin{aligned}\dot{\pi}^{ij} = -\alpha \frac{\delta \tilde{\mathcal{H}}}{\delta g_{ij}} &= -\alpha \bar{g} (R^{ij}(\bar{g}) - R(\bar{g})g^{ij}) - \alpha (2\pi^{ik}\pi^j_k - \pi\pi^{ij}) \\ &\quad + \bar{g} (\bar{\nabla}^i \bar{\nabla}^j \alpha - \bar{g}^{ij} \bar{\nabla}_k \bar{\nabla}^k \alpha) .\end{aligned}\quad (39)$$

Equation (39) for $\dot{\pi}^{ij}$ is the identity (31) with $\mathcal{R}^{ij} = 0$, which is equivalent to $R_{ij}(g) = 0$. Thus, (39) is a “strong” equation unlike its ADM counterpart, which requires in addition the imposition of a constraint: $\mathcal{H} = 0$.

In the present formulation, the canonical equations of motion hold everywhere on phase space with any parameter time t , a necessary condition for the issue of “constraint evolution” even to be *discussed* in the Hamiltonian framework. (See below in Sect. 4.)

If we define the “smeared” Hamiltonian as the integral of the Hamiltonian density,

$$\tilde{\mathcal{H}}_\alpha = \int d^3x' \alpha(t, x') \tilde{\mathcal{H}} , \quad (40)$$

the equation of motion for a general functional $F[\bar{g}, \pi; t, x)$ anywhere on the phase space is

$$\dot{F}[\bar{g}, \pi; t, x) = - \left\{ \tilde{\mathcal{H}}_\alpha, F \right\} + \tilde{\partial}_0 F , \quad (41)$$

where $\dot{(\cdot)}$ denotes our total time derivative and $\tilde{\partial}_0$ is a “partial” derivative of the form $\partial_t - \mathcal{L}_\beta$ acting only on explicit spacetime dependence. The Poisson bracket is

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta g_{ij}(t, x)} \frac{\delta G}{\delta \pi^{ij}(t, x)} - \frac{\delta G}{\delta g_{ij}(t, x)} \frac{\delta F}{\delta \pi^{ij}(t, x)} \right) , \quad (42)$$

and one sees that time evolution is generated by the Hamiltonian vector field

$$\begin{aligned}\mathcal{X}_{\tilde{\mathcal{H}}_\alpha} &= \int d^3x \left\{ \alpha(2\pi_{ij} - \pi g_{ij}) \frac{\delta}{\delta g_{ij}} - [\alpha \bar{g}(R^{ij}(\bar{g}) - R(\bar{g})g^{ij}) \right. \\ &\quad \left. + \alpha(2\pi^{ik}\pi^j_k - \pi\pi^{ij}) - \bar{g}(\bar{\nabla}^i \bar{\nabla}^j \alpha - g^{ij} \bar{\nabla}_k \bar{\nabla}^k \alpha)] \frac{\delta}{\delta \pi^{ij}} \right\} .\end{aligned}\quad (43)$$

Because it does not contain any explicit constraint dependence, (43) is a valid time evolution operator on the entire phase space. It is clear that the $(\dot{\bar{g}}, \dot{\pi})$ equations come from (43) applied to the canonical variables. The harmonic time slicing equation (23) results from application of (43) to N , and the wave equation for N comes from a repeated application of (43) to (23).

Evolution equations for the “constraints” are computed to be

$$\bar{\partial}_0 \tilde{\mathcal{H}} = - \left\{ \tilde{\mathcal{H}}_\alpha, \tilde{\mathcal{H}} \right\} = \alpha \bar{g} g^{ij} \partial_i \mathcal{H}_j + 2 \bar{g} g^{ij} \mathcal{H}_i \bar{\nabla}_j \alpha, \quad (44)$$

$$\bar{\partial}_0 \mathcal{H}_j = - \left\{ \tilde{\mathcal{H}}_\alpha, \mathcal{H}_j \right\} = \alpha \partial_j \tilde{\mathcal{H}} + 2 \tilde{\mathcal{H}} \partial_j \alpha, \quad (45)$$

where $\bar{\nabla}_j \alpha = \partial_j \alpha + \alpha \bar{g}^{-1/2} \partial_i \bar{g}^{1/2}$. These are well-posed evolution equations for the constraints, and they are equivalent to the twice-contracted Bianchi identities when $\mathcal{R}_{ij} = 0$ or $R_{ij} = 0$ (see below).

These results shed new light on the Dirac “algebra” of constraints [36]. It is well known that the Dirac algebra is not the spacetime diffeomorphism algebra. This can be seen from the fact that while the action (36) is invariant under transformations generated by \mathcal{H}_j and $\tilde{\mathcal{H}}$, [37] the equations of motion that follow from this action are $R_{ij}(g) = 0$ even when \mathcal{H}_j and $\tilde{\mathcal{H}}$ do not vanish. These equations of motion are preserved by spatial diffeomorphisms and time translations along their flow in phase space, whereas a general spacetime diffeomorphism applied to $R_{ij}(g) = 0$ would mix in the constraints.

A second important view of the Dirac algebra results from the direct and beautiful dynamical meaning of its once-smeared form. Equations (44) and (45) express *consistency* of the constraints as a *well posed* initial-value problem. If the constraint functions vanish in some region on an initial time slice, they continue to do so under evolution by the Hamiltonian vector field into the domain of dependence of that initial region. This mechanism follows from the dual role of $\tilde{\mathcal{H}}$ as a constraint and as part of the generator of time translations of functionals of the canonical variables anywhere on the phase space.

Let us take note that the Hamiltonian constraint *per se* does not express the dynamics of the theory; the equation of dynamics is (41). In its “altered” role, the Hamiltonian constraint function simply vanishes as an initial value condition, from which $\bar{g}^{1/2}$ is determined as in the initial value problem. [30] Then N can be constructed from α . The Hamiltonian constraint, once solved, remains so according to the results embodied in (44) and (45).

4 Contracted Bianchi Identities

The results on canonical dynamics that follow on using α as an undetermined multiplier are also reflected in the manner in which the twice-contracted

Bianchi identities,

$$\nabla_\beta G^\beta{}_\alpha \equiv 0 , \quad (46)$$

can be written as a first-order symmetrizable hyperbolic system [29]. (In the absence of hyperbolic form, (46) is practically useless in providing physical equations of motion for the constraints when they are *not* satisfied.) Likewise, this system extends to matter (see below). (Frittelli obtained well-posedness for (46) by other methods [38].)

We recall that the equations of motion of the canonical momenta in vacuum are

$$\mathcal{R}_{ij} \equiv R_{ij}(g) - g_{ij}R^k{}_k(g) = 0 , \quad (47)$$

while the weight zero Hamiltonian constraint is

$$C = 2G^0{}_0(g) = K_{ij}K^{ij} - K^2 - R(\bar{g}) = 0 , \quad (48)$$

and the weight zero one-form momentum constraint is

$$C_i = 2NR^0{}_i(g) = 2\bar{\nabla}^j(K_{ij} - Kg_{ij}) = 0 . \quad (49)$$

Recall the identity (33):

$$G_{ij}(g) + g_{ij}G^0{}_0(g) \equiv R_{ij}(g) - g_{ij}R^k{}_k(g) \equiv \mathcal{R}_{ij} . \quad (50)$$

Combining (47), (48), (49), and (33) with (46) gives the twice-contracted Bianchi identities as a FOSH system

$$\dot{C} - N\bar{\nabla}^j C_j \equiv 2(C_j \bar{\nabla}^j N + NKC - NK^{ij}[\mathcal{R}_{ij}]) , \quad (51)$$

$$\dot{C}_j - N\bar{\nabla}_j C \equiv 2\left(C\bar{\nabla}_j N + \frac{1}{2}NKC_j - \bar{\nabla}^i(N[\mathcal{R}_{ij}])\right) . \quad (52)$$

Substituting $\mathcal{H}_i = \bar{g}^{1/2}C_i$, $\tilde{\mathcal{H}} = \bar{g}C$, and setting the equations of motion $\mathcal{R}_{ij} = 0$ in (51) and (52) yields the evolution equations of the unsmeared constraints as in (44) and (45).

Similar considerations show how to put the “matter conservation” equations $\nabla_\beta T^{\alpha\beta} = 0$ into well-posed form. This was also carried out by one of us (YCB) and Noutchegueme [39] but the results obtained here are more immediately physical. Unlike [39], we use the energy density $\varepsilon = -T^0{}_0$ rather than $\rho^{00}(\rho^\alpha{}_\beta \equiv T^\alpha{}_\beta - \frac{1}{2}\delta^\alpha{}_\beta T^\mu{}_\mu)$ to obtain this result. It is clear that such a result is possible because

$$H_{\alpha\beta} \equiv \kappa^{-1}G_{\alpha\beta}(g) - T_{\alpha\beta} \quad (53)$$

vanishes as Einstein's equation and in any case satisfies $\nabla_\beta H^\beta_\alpha = 0$. We can treat $H_{\alpha\beta}$ as we did $G_{\alpha\beta}$ above. ($\kappa = 8\pi G$; $c = 1$.) The result is nevertheless of interest as it presents the continuity and relativistic Euler equations of matter in a well-posed form.

Straightforwardly expanding $\nabla_\beta T^\beta_0 = 0$ and $\nabla_\beta T^\beta_i = 0$ gives the continuity and Euler equations (cf. [30], p. 89), with $\varepsilon \equiv -T^0_0$ and the matter current one-form $j_i \equiv NT^0_i$. The continuity equation is

$$\bar{\partial}_0 \varepsilon + N \bar{\nabla}^i j_i = N(K_{ij} T^{ij} + K\varepsilon - 2j_i a^i), \quad (54)$$

where $a_i \equiv \bar{\nabla}_i \log N$ is the acceleration of observers at rest in a given time-slice. Likewise, we find for Euler's equation

$$\bar{\partial}_0 j_i + N \bar{\nabla}_j T^j_i = N(K^j_i - T^j_i a_j - \varepsilon a_i). \quad (55)$$

The divergence term on the left side of (55) spoils the well-posed FOSH form we seek. However, if we use the identity $G_{ij}(g) + g_{ij}G^0_0(g) \equiv R_{ij}(g) - g_{ij}R^k_k(g)$ and the Einstein equations $\kappa^{-1}G_{\alpha\beta} - T_{\alpha\beta} = 0$, or $\kappa^{-1}R_{\alpha\beta} = \rho_{\alpha\beta}$, we obtain ($\varepsilon = -T^0_0$, $j_i = NT^0_i$)

$$T^j_i - \delta^j_i \varepsilon = (\rho^j_i - \delta^j_i \rho^k_k) \equiv \mathcal{S}^j_i. \quad (56)$$

Then (54) and (55) obtain well-posed form (if \mathcal{S}^j_i is assumed known),

$$\bar{\partial}_0 \varepsilon + N \bar{\nabla}^i j_i = -2j_i \bar{\nabla}^i N + 2NK\varepsilon + NK^{ij} \mathcal{S}_{ij}, \quad (57)$$

$$\bar{\partial}_0 j_i + N \bar{\nabla}_i \varepsilon = -2\varepsilon \bar{\nabla}_i N + NKj_i - \bar{\nabla}^j (N \mathcal{S}_{ij}). \quad (58)$$

By combining (57) plus (49), and (58) plus (50), we obtain expressions of gravity constraint evolution in the presence of matter,

$$\bar{\partial}_0 C^T - N \bar{\nabla}^i C^T_i = 2(C^T_j \bar{\nabla}^j N + NK C^T - NK^{ij}(\kappa^{-1} \mathcal{R}_{ij} - \mathcal{S}_{ij})) , \quad (59)$$

$$\bar{\partial}_0 C^T_j - N \bar{\nabla}_j C^T = 2 \left[C^T \bar{\nabla}_j N + \frac{1}{2} NK C^T_j - \bar{\nabla}^i (N(\kappa^{-1} \mathcal{R}_{ij} - \mathcal{S}_{ij})) \right] \quad (60)$$

where $C^T \equiv C + 2\varepsilon$ and $C^T_j = C_j - 2j_j$. This is just the form we would anticipate on the basis of Hamiltonian dynamics and the form (49) and (50) of the vacuum constraints. Thus, for gravity plus a matter field, we obtain results analogous to (44) and (45) for the total system. If there are no violations of constraints, then $C^T = 0$, $C^T_j = 0$, while the dynamical gravity equation is $\kappa^{-1} \mathcal{R}_{ij} - \mathcal{S}_{ij} = 0$.

5 Wave Equation for K_{ij}

Einstein's equations, viewed mathematically as a system of second-order partial differential equations for the metric, do not form a hyperbolic system without modification and are not manifestly well-posed, though, of course, physical information does propagate at the speed of light. A well-posed hyperbolic system admits unique solutions depending continuously on the initial data and seems to be required for robust, stable numerical integration and for full treatment by the methods of modern analysis, for example, exploitation of energy estimates. The well-known traditional approach achieves hyperbolicity through special coordinate choices.³ The formulation described here permits coordinate gauge freedom. Because these exact nonlinear theories incorporate the constraints, they are natural starting points for developing gauge-invariant perturbation theory.

Consider a globally hyperbolic manifold $V = \Sigma \times \mathbb{R}$ with the metric as given in the introduction. To achieve hyperbolicity for the $3 + 1$ equations, we proceed as follows.

By taking a time derivative of $R_{ij}(g)$ and subtracting appropriate spatial covariant derivatives of the momentum constraints, one of us (YCB) and T. Ruggeri [18] (see also [19], where the shift is not set to zero) obtained an equation with a wave operator acting on the extrinsic curvature. In vacuum, one finds

$$\bar{\partial}_0 R_{ij}(g) - \bar{\nabla}_i R_{0j} - \bar{\nabla}_j R_{0i} = N \bar{\square}_g K_{ij} + J_{ij} + S_{ij} = 0, \quad (61)$$

where $\bar{\square}_g = -(N^{-1} \bar{\partial}_0)^2 + \bar{\nabla}_k \bar{\nabla}^k$, J_{ij} consists of terms at most first order in derivatives of K_{ij} , second order in derivatives of g_{ij} , and second order in derivatives of N , and

$$S_{ij} = -N^{-1} \bar{\nabla}_i \bar{\nabla}_j (\bar{\partial}_0 N + N^2 K). \quad (62)$$

The term S_{ij} is second order in derivatives of K_{ij} and would spoil hyperbolicity of the wave operator $\bar{\square}$ acting on K_{ij} . Hyperbolicity is achieved by setting $N = \bar{g}^{1/2} \alpha(t, x)$, or

$$\bar{\partial}_0 N + N^2 K = \bar{g}^{1/2} \bar{\partial}_0 \alpha(t, x), \quad (63)$$

³The classic second-order fully harmonic form was given in [6, 7] and discussed, for example, in [9]. It will not be discussed in this article. A FOSH form based on these equations was given first by Fischer and Marsden in [23].

as discussed in Sect. 2. The resulting equation combined with (28) forms a quasi-diagonal hyperbolic system for the metric g_{ij} with principal operator $\bar{\partial}_0 \bar{\square}$. This system can also be put in first order symmetric hyperbolic form [19, 11], by the introduction of sufficient auxiliary variables and by use of the equation for R_{00} (thus incorporating the Hamiltonian constraint). The Cauchy data for the system (in vacuum) [18, 19] are (1) (\bar{g}, K) such that the constraints $R_{0i} = 0$, $G^0_0 = 0$ hold on the initial slice; (2) $\bar{\partial}_0 K_{ij}$ such that $R_{ij} = 0$ on the initial slice; and (3) $N > 0$ arbitrary on the initial slice. Note that the shift $\beta^k(x, t)$ is arbitrary. Using the Bianchi identities, one can prove [18, 19] that this system is fully equivalent to the Einstein equations. The point is that quasi-diagonal Leray [40] hyperbolic systems have well posed Cauchy problems and therefore unique solutions for given initial data. Because every solution of the Einstein equations also satisfies the $\bar{\square} K_{ij}$ equation in particular and provides initial data for it, uniqueness implies, conversely, that if the initial data for the $\bar{\square} K_{ij}$ equation are Einsteinian, all solutions of Einstein's equations, and only these, are captured. The restriction on the initial value of $\bar{\partial}_0 K_{ij}$ prevents the higher derivative from introducing spurious unphysical solutions.

All variables propagate either with characteristic speed zero or the speed of light. The only variables which propagate at the speed of light have the dimensions of curvature, and one sees that this is a theory of propagating curvature. However, a FOSH system that propagates curvature is more transparent in the ‘‘Einstein-Bianchi’’ form (next section).

In the above formulation, the shift and α are arbitrary. This and our other systems (except the Einstein-Christoffel system in Sect. 7) are manifestly spatially covariant and all time slicings (using α) are allowed. Spacetime covariance is therefore present, but not completely manifest.

By taking another time derivative and adding an appropriate derivative of R_{00} , one finds (in vacuum) [29]

$$\bar{\partial}_0 \bar{\partial}_0 R_{ij} - \bar{\partial}_0 \bar{\nabla}_i R_{0j} + \bar{\partial}_0 \bar{\nabla}_j R_{0i} + \bar{\nabla}_i \bar{\nabla}_j R_{00} = \bar{\partial}_0 (N \bar{\square} K_{ij}) + \mathcal{J}_{ij} = 0, \quad (64)$$

where \mathcal{J}_{ij} consists of terms at most third order in derivatives of g_{ij} and second order in derivatives of K_{ij} . Together with $\bar{\partial}_0 g_{ij}$, these form a system for (\bar{g}, K) which is hyperbolic non-strict in the sense of Leray-Ohya. [41] Here, the lapse itself, as well as the shift, is arbitrary ($N > 0$). The Cauchy data of the previous form (in vacuum) must be supplemented by $\bar{\partial}_0 \bar{\partial}_0 K_{ij}$ such that $\bar{\partial}_0 R_{ij} = 0$ on the initial slice. This guarantees that the system is fully

equivalent to Einstein's theory (except that its solutions are not in Sobolev spaces [29]). This system does not have a first order symmetric hyperbolic formulation, but has been used very effectively in perturbation theory [42] and in other applications [43, 44].

6 Einstein-Bianchi Hyperbolic System

To obtain a first order symmetric hyperbolic system, one can use the Riemann tensor of the spacetime metric. It satisfies the Bianchi identities for the spacetime geometry

$$\nabla_\alpha R_{\beta\gamma\lambda\mu} + \nabla_\beta R_{\gamma\alpha\lambda\mu} + \nabla_\gamma R_{\alpha\beta\lambda\mu} \equiv 0 . \quad (65)$$

These identities imply by contraction and use of the symmetries of the Riemann tensor

$$\nabla_\alpha R^\alpha_{\mu\beta\gamma} + \nabla_\gamma R_{\beta\mu} + \nabla_\beta R_{\gamma\mu} \equiv 0 . \quad (66)$$

If the Ricci tensor $R_{\alpha\beta}$ satisfies the Einstein equations ($\kappa = c = 1$)

$$R_{\alpha\beta} = \rho_{\alpha\beta} , \quad (67)$$

then the previous identities imply the equations

$$\nabla_\alpha R^\alpha_{\mu\beta\gamma} = \nabla_\beta \rho_{\gamma\mu} - \nabla_\gamma \rho_{\beta\mu} . \quad (68)$$

The first equations with $(\alpha\beta\gamma) = (ijk)$ and the last one with $\mu = 0$ do not contain derivatives of the Riemann tensor transverse to M_t . They are considered as “constraints” and will be identically satisfied (initially) in our method. They remain satisfied in an exact integration. All detail and rigor concerning this elegant system is given in [22, 51], to which the reader is referred. It has 66 equations, just as do the Einstein-Ricci first order curvature equations.

The system we are now developing [21] is similar to an analogous system obtained by H. Friedrich [25] that is based on the Weyl tensor. The Weyl tensor system is causal but with additional unphysical characteristics.

We wish first to show that the remaining equations are, for $n = 3$ in the vacuum case, when g is given, a symmetric first order hyperbolic system for the double two-form $R_{\alpha\beta\lambda\mu}$. For this purpose, following Bel [52, 53] we

introduce two pairs of “electric” and “magnetic” space tensors associated with a spacetime double two-form A ,

$$N^2 E_{ij}(g) \equiv A_{0i0j} \quad (69)$$

$$D_{ij}(g) \equiv \frac{1}{4} \epsilon_{ihk} \epsilon_{jlm} A^{hklm} \quad (70)$$

$$NH_{ij}(g) \equiv \frac{1}{2} \epsilon_{ihk} A^{hk}{}_{0j} \quad (71)$$

$$NB_{ji}(g) \equiv \frac{1}{2} A_{0j}{}^{hk} \epsilon_{ihk} \quad (72)$$

where ϵ_{ijk} is the volume form of \bar{g} . It results from the symmetry of the Riemann tensor R with respect to its first and second pairs of indices (R is a “symmetric double two-form”) that if $A \equiv R$, then E and D are symmetric while $H_{ij} = B_{ji}$. A useful identity for a symmetric double two-form like R , with a tilde representing the spacetime double dual, is (“Lanczos identity”)

$$\tilde{R}_{\alpha\beta\lambda\mu} + R_{\alpha\beta\lambda\mu} = C_{\alpha\lambda} g_{\beta\mu} - C_{\alpha\mu} g_{\beta\lambda} + C_{\beta\mu} g_{\alpha\lambda} - C_{\beta\lambda} g_{\alpha\mu} , \quad (73)$$

where $C_{\alpha\beta} = R_{\alpha\beta} - (1/4)g_{\alpha\beta}R$. It follows that when $R_{\alpha\beta} = \lambda g_{\alpha\beta}$, then $E = -D$ and $H = B$. In order to avoid introducing unphysical characteristics, and to be able to extend the treatment to the non-vacuum case, *we do not use these properties in the evolution equations*, but write them as a first order system for an arbitrary double two-form A , as follows:

$$\nabla_0 A_{hk0j} + \nabla_k A_{0h0j} - \nabla_h A_{0k0j} = 0 , \quad (74)$$

$$\nabla_0 A^0{}_{i0j} + \nabla_h A^h{}_{i0j} = \nabla_0 \rho_{ji} - \nabla_j \rho_{0i} , \quad (75)$$

and analogous equations with the pair $(0j)$ replaced by (lm) . One obtains a first order system for the unknowns E , H , D , and B by using the relations inverse to the definitions above. The principal parts of these equations, all with one definite index fixed on E , H , D , and B , are identical to the corresponding Maxwell equations. The characteristic matrix of this “Maxwell” part of the system has determinant $-N^6(\xi_0\xi^0)(\xi_\alpha\xi^\alpha)^2$. The system obtained has a principal matrix consisting of 6 identical 6 by 6 blocks around the diagonal, which are symmetrizable and hyperbolic. Hence, the system is symmetric hyperbolic, when g is a given metric such that \bar{g} is properly riemannian and $N > 0$.

To relate the Riemann tensor to the metric \bar{g} we use the definition

$$\bar{\partial}_0 g_{ij} = -2NK_{ij} \quad (76)$$

and we use the $3 + 1$ identities given in the Introduction. *Note that in this section all Γ 's are spatial.*

We next choose $N = \bar{g}^{1/2}\alpha(t, x)$. We generalize somewhat the ideas used by Friedrich (see [25]) for the Weyl tensor to write a symmetric hyperbolic system for K and Γ , namely we obtain equations relating Γ and K , for a given double two-form A , and by considering the definition of K and the $3 + 1$ decomposition of the Riemann tensor, replacing in these identities the Riemann tensor by A . To deduce from this system a symmetric hyperbolic first order system, with the algebraic form of the harmonic gauge $N = \bar{g}^{1/2}\alpha$, one uses the fact that in this gauge one has

$$\Gamma^h_{ih} = \partial_i \log N - \partial_i \log \alpha . \quad (77)$$

We obtain

$$\begin{aligned} \bar{\partial}_0 \Gamma^h_{ij} + N \bar{\nabla}^h K_{ij} &= N K_{ij} g^{hk} (\Gamma^m_{mk} + \partial_k \log \alpha) \\ &\quad - 2N K^h_{(i} (\Gamma^m_{j)m} + \partial_j \log \alpha) \\ &\quad - N (\epsilon^k_{(j}{}^h B_{i)k} + H_{k(i} \epsilon^k_{j)}{}^h) , \end{aligned} \quad (78)$$

and

$$\begin{aligned} \bar{\partial}_0 K_{ij} + N \partial_h \Gamma^k_{ij} &= N [\Gamma^m_{ih} \Gamma^h_{jm} - (\Gamma^h_{ih} + \partial_i \log \alpha) (\Gamma^k_{jk} + \partial_j \log \alpha)] \\ &\quad - N (\partial_i \partial_j \log \alpha - \Gamma^k_{ij} \partial_k \log \alpha) \\ &\quad - N [D_{(ij)} + E_{(ij)} - D^k_{k} g_{ij} - K K_{ij}] . \end{aligned} \quad (79)$$

The system obtained for K and Γ has a characteristic matrix composed of 6 blocks around the diagonal, each block a 4 by 4 matrix that is symmetrizable hyperbolic, with characteristic polynomial $-N^4(\xi_0 \xi^0)(\xi_\alpha \xi^\alpha)$.

The whole system for A , K , Γ , \bar{g} is symmetrizable hyperbolic, with characteristics the light cone and the normal to M_t . It is somewhat involved to prove that a solution of the constructed system satisfies the Einstein equations if the initial data satisfy the constraints, but we can argue as follows. We consider the vacuum case with initial data satisfying the Einstein constraints. These initial data determine the initial values of Γ , and also, if β and N are known at $t = 0$, the initial values of A_{ijhm} , A_{jih0} , A_{i0jh} by using the decomposition formulas. (We set A equal to the Riemann tensor on the initial surface.) We use the Lanczos formula to determine A_{i0j0} initially. We know that our symmetrizable hyperbolic system has one and only one solution. Because a solution of Einstein's equations with $N = \bar{g}^{1/2}\alpha$, proved to

exist in the section on $\bar{\square}K_{ij}$, satisfies together with its Riemann tensor the present system and takes the same initial values, that solution coincides with the solution of the present system in their common domain of existence.

7 Einstein-Christoffel System

The first-order form of the wave equation for K_{ij} , the Einstein-Ricci system [19, 11], to which we have alluded, has 66 equations, the correct number for a curvature system as does the Einstein-Bianchi system of the previous section. It is symmetric hyperbolic and therefore well-posed. But it is natural to ask whether there is a simpler first-order system of fewer variables that is perhaps closer in form to (28) and (29). Frittelli and Reula [26, 27] proposed such a system, but their system has unphysical characteristics and is not written fully in terms of geometric variables. Here we deduce a different system having only physical characteristics and expressed in geometric variables. We understand (private communication to JWY) that James Bardeen has likewise obtained a similar system improving that found in [16, 17]. While our derivation [10] can proceed systematically by direct construction of an energy norm and the characteristic speeds, a more heuristic derivation is indicated here from the structure of the wave equation for K_{ij} . Not that in this section all Γ 's are spatial.

In the dynamical spacetime Ricci tensor $R_{ij}(g)$, (29), one has a dynamical equation for the extrinsic curvature K_{ij} in terms of spatial derivatives of the spatial Christoffel symbols. When the lapse N is replaced by the slicing density through $\alpha\bar{g}^{1/2}$, the differentiated Christoffel terms become

$$\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} - \partial_i \Gamma^k_{jk} . \quad (80)$$

This may be read as the divergence of a linear combination of Christoffel symbols, which puts the $R_{ij}(\bar{g})$ equation in a form reminiscent of the structure of one of the first-order equations for a free wave, namely

$$\partial_0 u + \partial^k v_k = 0 . \quad (81)$$

We would have a symmetric hyperbolic system if there were an analog of the other equation for a wave,

$$\partial_0 v_k + \partial_k u = 0 . \quad (82)$$

Some manipulation quickly leads to the conclusion that K_{ij} and the Christoffel combination above are not paired in a symmetric hyperbolic system like u and v_k .

Recall however, that the free wave equation $(\partial_0)^2 u - \partial^k \partial_k u = 0$ is obtained by taking a time derivative of (81) and subtracting the divergence of (82). In obtaining the wave equation for K_{ij} (61) we have taken a time derivative of R_{ij} and subtracted a (suitably symmetrized) divergence of the momentum constraint R_{0i} . This motivates the speculation that in gravity the “other” equation should be related to the momentum constraint and its sole spatial derivative should be $\partial_k K_{ij}$.

Following this idea about the second equation leads one to consider an equation of the form

$$g_{ki} R_{j0} + g_{kj} R_{i0} = -\bar{\partial}_0 f_{kij} - \partial_k (N K_{ij}) + l.o._{kij} \quad (83)$$

where $l.o._{kij}$ are lower order terms involving no derivatives of f_{kij} or K_{ij} . One must choose f_{kij} from a linear combination of spatial derivatives of the metric. Introduce

$$\mathcal{G}_{kij} = \partial_k g_{ij} \quad (84)$$

and use the identity

$$\bar{\partial}_0 (\partial_k g_{ij}) = -\partial_k (2N K_{ij}) \quad (85)$$

to find that

$$f_{kij} = \frac{1}{2} \mathcal{G}_{kij} - g_{k(i} g^{rs} (\mathcal{G}_{|rs|j)} - \mathcal{G}_{j)rs}) \quad (86)$$

produces the correct coordinate derivatives occurring in the momentum constraints. The lower order terms are those terms necessary to complete (83) into an identity and take the form

$$\begin{aligned} l.o._{kij} = & 2N K_{k(i} g^{rs} (\mathcal{G}_{|rs|j)} - \mathcal{G}_{j)rs}) \\ & + 2g_{k(i} \left[K_{j)m} \partial^m N - K \partial_j N \right. \\ & \left. + N K_{j)m} g^{rs} \Gamma_{rs}^m(\mathcal{G}) + \frac{1}{2} N (\mathcal{G}_{j)rs} - 2\mathcal{G}_{|rs|j)} K^{rs} \right], \end{aligned} \quad (87)$$

where the spatial Christoffel symbols are constructed from \mathcal{G}_{ijk} ,

$$\Gamma_{kij}(\mathcal{G}) \equiv (1/2) (\mathcal{G}_{jki} + \mathcal{G}_{ikj} - \mathcal{G}_{kij}) . \quad (88)$$

(It is clear that only one of the three-index symbols \mathcal{G}_{kij} , Γ_{kij} , and f_{kij} is needed, say f_{kij} . The necessary algebra will not be reproduced here.)

One then easily verifies that by expressing (88) in terms of derivatives of the metric (assuming a metric compatible connection), we can manipulate it to take the form of a divergence of f_{kij} plus lower order terms. The dynamical Ricci equation becomes

$$R_{ij} = -N^{-1}\bar{\partial}_0 K_{ij} - \partial^k f_{kij} + l.o._{ij} , \quad (89)$$

where

$$\begin{aligned} l.o._{ij} = & K K_{ij} - 2K_{ik} K^k_j - \alpha^{-1} (\partial_i \partial_j - \Gamma^k_{ij}(\mathcal{G}) \partial_k) \alpha \\ & - (\Gamma^k_{ki}(\mathcal{G}) + \alpha^{-1} \partial_i \alpha) (\Gamma^m_{mj}(\mathcal{G}) + \alpha^{-1} \partial_j \alpha) \\ & + 2\Gamma^k_{mk}(\mathcal{G}) \Gamma^m_{ij}(\mathcal{G}) - \Gamma^k_{mj}(\mathcal{G}) \Gamma^m_{ik}(\mathcal{G}) \\ & + g^{kr} g^{sm} [\mathcal{G}_{krs} f_{mij} + \mathcal{G}_{km(i} \mathcal{G}_{j)rs} - \mathcal{G}_{krs} \mathcal{G}_{(ij)m}] . \end{aligned} \quad (90)$$

Together with (28), (83) and (89) constitute a symmetric hyperbolic system for the evolution of g_{ij} , K_{ij} and f_{kij} .

Note that once f_{kij} or, equivalently, \mathcal{G}_{kij} are introduced as variables, the relation (88) becomes an initial condition and does not *a priori* hold for all time. Equation (83) can be related to metric compatibility by putting it in the form

$$\begin{aligned} 4g_{k(i} R_{j)0} = & \bar{\partial}_0 \mathcal{G}_{kij} + \partial_k (2N K_{ij}) \\ & - 4g_{k(i} N \bar{\nabla}^m (K_{j)m} - g_{j)m} K) . \end{aligned} \quad (91)$$

Here, one sees that if the momentum constraint is satisfied for all time, then

$$\mathcal{G}_{kij} = 2\Gamma_{(ij)k} = \partial_k g_{ij} \quad (92)$$

and the connection is metric compatible. If the momentum constraint is violated, metric compatibility is sacrificed. This shows the price paid to achieve a symmetric hyperbolic system that is close to the canonical equations: the momentum constraints become dynamical, and metric compatibility is lost if the latter are violated.

8 Conformal “Thin Sandwich” Data for the Initial Value Problem

The standard approach to the initial value problem is the “conformal method,” the fundamental rudiments of which were introduced by Lichnerowicz [8].

The essentially complete form was developed by two of us (YCB and JWY), see [34, 9]. Basic theorems were obtained by us and by O’Murchadha [45, 35, 46], and Isenberg and Moncrief [47]. The older approach concentrates (in the vacuum case to which we restrict ourselves) on the construction of the spatial metric $g_{ij} = \psi^4 \gamma_{ij}$ and the traceless part $A^{ij} = \psi^{-10} \lambda^{ij}$ of the extrinsic curvature, where $K_{ij} = A_{ij} + \frac{1}{3} g_{ij} K$. Here, γ_{ij} is a proper riemannian metric given freely, λ^{ij} is constructed by a tensor decomposition method [34, 48], and K is given freely (not conformally transformed). Note that one may as well assume $\det(\gamma_{ij}) = 1$ because only the conformal equivalence class of the metric matters: the entire method is “conformally covariant.”

N.B.: In Sect. 8, only spatial metrics will be used. Therefore, *all overbars are dropped in this final section.*

Here, we discuss a new interpretation of the four Einstein vacuum initial-value constraints. (The presence of matter would add nothing new to the analysis.) Partly in the spirit of a “thin sandwich” viewpoint, this approach is based on prescribing the *conformal* metric [1] on each of two nearby space-like hypersurfaces (“time slices” $t = t'$ and $t = t' + \delta t$) that make a “thin sandwich” (TS). *Essential use is made of the understanding of the slicing function in general relativity.* The new formulation could prove useful both conceptually and in practice, as a way to construct initial data in which one has a hold on the input data different from that in the currently accepted approach. The new approach allows us to *derive* from its dynamical and metrical foundations the important scaling law $A^{ij} = \psi^{-10} \lambda^{ij}$ for the traceless part of the extrinsic curvature. This rule is simply postulated in the one-hypersurface approach.

The constraint equations on Σ are, in vacuum,

$$\nabla_j (K^{ij} - K g^{ij}) = 0 , \quad (93)$$

$$R(g) - K_{ij} K^{ij} + K^2 = 0 \quad (94)$$

where $R(g)$ is the spatial scalar curvature of g_{ij} , ∇_j is the Levi-Civita connection of g_{ij} ; and K is the trace of K_{ij} , also called the “mean curvature” of the slice.

The time derivative of the spatial metric g_{ij} is related to K_{ij} , N , and the shift vector β^i by

$$\partial_t g_{ij} \equiv -2N K_{ij} + (\nabla_i \beta_j + \nabla_j \beta_i) , \quad (95)$$

where $\beta_j = g_{ji}\beta^i$. The fixed spatial coordinates x of a point on the “second” hypersurface, as evaluated on the “first” hypersurface, are displaced by $\beta^i(x)\delta t$ with respect to those on the first hypersurface, with an orthogonal link from the first to the second surface as a fiducial reference: $\beta_i = \frac{\partial}{\partial t} * \frac{\partial}{\partial x^i}$, where $*$ is the physical spacetime inner product of the indicated natural basis four-vectors. The essentially arbitrary direction of $\frac{\partial}{\partial t}$ is why $N(x)$ and $\beta^i(x)$ appear in the TS formulation. In contrast, the tensor K_{ij} is always determined by the behavior of the unit normal on one slice and therefore does not possess the kinematical freedom, i.e., the gauge variance, of $\frac{\partial}{\partial t}$. Therefore, N and β^i do not appear in the one-hypersurface IVP for (Σ, g, K) .

Turning now to the conformal metrics in the IVP, we recall that two metrics g_{ij} and γ_{ij} are conformally equivalent if and only if there is a scalar $\psi > 0$ such that $g_{ij} = \psi^4 \gamma_{ij}$. The conformally invariant representative of the entire conformal equivalence class, in three dimensions, is the weight $(-2/3)$ unit-determinant “conformal metric” $\hat{g}_{ij} = g^{-1/3} g_{ij} = \gamma^{-1/3} \gamma_{ij}$ with $g = \det(g_{ij})$ and $\gamma = \det(\gamma_{ij})$. Note particularly that for any small perturbation, $g^{ij} \delta \hat{g}_{ij} = 0$. We will use the important relation

$$g^{ij} \partial_t \hat{g}_{ij} = \gamma^{ij} \partial_t \hat{g}_{ij} = \hat{g}^{ij} \partial_t \hat{g}_{ij} = 0 . \quad (96)$$

In the following, rather than use the mathematical apparatus associated with conformally weighted objects such as \hat{g}_{ij} , we find it simpler to use ordinary scalars and tensors to the same effect. Thus, let the role of \hat{g}_{ij} on the first surface be played by a given metric γ_{ij} such that the physical metric that satisfies the constraints is $g_{ij} = \psi^4 \gamma_{ij}$ for some scalar $\psi > 0$. (This corresponds to “dressing” the initial unimodular conformal metric \hat{g}_{ij} with the correct determinant factor $g^{1/3} = \psi^4 \gamma^{1/3}$. This process does not alter the conformal equivalence class of the metric.) The role of the conformal metric on the second surface is played by the metric $\gamma'_{ij} = \gamma_{ij} + u_{ij} \delta t$, where, in keeping with (96), the velocity tensor $u_{ij} = \partial_t \gamma_{ij}$ is chosen such that

$$\gamma^{ij} u_{ij} = \gamma^{ij} \partial_t \gamma_{ij} = 0 . \quad (97)$$

Then, to first order in δt , γ'_{ij} and γ_{ij} have equal determinants, as desired; but γ_{ij} and γ'_{ij} are not in the same conformal equivalence class in general.

We now examine the relation between the covariant derivative operators D_i of γ_{ij} and ∇_i of g_{ij} . The relation is determined by

$$\Gamma^i_{jk}(g) = \Gamma^i_{jk}(\gamma) + 2\psi^{-1} (2\delta^i_{(j} \partial_{k)} \psi - \gamma^{il} \gamma_{jk} \partial_l \psi) , \quad (98)$$

from which follows the scalar curvature relation first used in an initial-value problem by Lichnerowicz [5],

$$R(g) = \psi^{-4}R(\gamma) - 8\psi^{-5}\Delta_\gamma\psi, \quad (99)$$

where $\Delta_\gamma \equiv \gamma^{kl}D_kD_l\psi$ is the scalar Laplacian associated with γ_{ij} .

Next, we solve (95) for its traceless part

$$\partial_t g_{ij} - \frac{1}{3}g_{ij}g^{kl}\partial_t g_{kl} \equiv V_{ij} = -2NA_{ij} + (L_g\beta)_{ij} \quad (100)$$

with $A_{ij} \equiv K_{ij} - (1/3)Kg_{ij}$ and

$$(L_g\beta)_{ij} \equiv \nabla_i\beta_j + \nabla_j\beta_i - (2/3)g_{ij}\nabla^k\beta_k. \quad (101)$$

Expression (101) vanishes, for non-vanishing β^i , if and only if g_{ij} admits a conformal Killing vector $\beta^i = k^i$. Clearly, k^i would also be a conformal Killing vector of γ_{ij} , or of any metric conformally equivalent to g_{ij} , with no scaling of k^i . This teaches us that β^i does not scale. That β^i does not scale also follows because, as generator of a spatial diffeomorphism, it is not a dynamical variable. We take the latter “rule” as a matter of principle.

It is clear in (100) that the left hand side u_{ij} satisfies $u_{ij} = \psi^4 V_{ij}$ because the terms in $\dot{\psi}$ cancel out. Furthermore, a straightforward calculation shows that

$$(L_g\beta)_{ij} = \psi^4 [L_\gamma(\psi^{-4}\beta)]_{ij}; \quad (L_g\beta)^{ij} = \psi^4 (L_\gamma\beta)^{ij}, \quad (102)$$

where $\psi^{-4}\beta_j = \gamma_{ij}\beta^i$. Next, we note that the lapse function N has essential non-trivial conformal behavior. This is a new element in the IVP analysis. The slicing function $\alpha(t, x) > 0$ can replace the lapse function N ,

$$N = g^{1/2}\alpha. \quad (103)$$

(The treatment here extends the one in [49] in a simple but interesting way.) We have concluded that α is not a dynamical variable and therefore does not scale. Furthermore, without loss of generality, we can set $\det(\gamma_{ij}) = 1$ and thus $g^{1/2} = \psi^6$. Then

$$N = \psi^6\alpha. \quad (104)$$

Finally, we fix K and require that it does not scale, as in the standard treatment of the IVP [50]. This step is absolutely essential for geometric consistency, as we shall see.

Next we solve (98) for A^{ij} , using the scaling rules established above, and find

$$A^{ij} = \psi^{-10} \left\{ \frac{1}{2\alpha} [(L_\gamma \beta)^{ij} - u^{ij}] \right\} . \quad (105)$$

The momentum constraint becomes

$$D_j \left[\frac{1}{2\alpha} (L_\gamma \beta)^{ij} \right] = D_j \left[\frac{1}{2\alpha} u^{ij} \right] + \frac{2}{3} \psi^6 \gamma^{ij} \partial_j K , \quad (106)$$

while the Hamiltonian constraint becomes [34]

$$8\Delta_\gamma \psi - R(\gamma)\psi + (\gamma_{ik}\gamma_{jl})A^{ij}A^{kl}\psi^{-7} - (2/3)K^2\psi^5 = 0 . \quad (107)$$

The unknowns (ψ, β) obey equations of the same form as do the conformal scalar potential ϕ and the vector potential W^i in the standard analysis [34, 9], but *no tensor splittings* are required. Further, (106) and (107) are coupled in only one direction when $K = \text{const}$.

Now we note two interesting consequences of this approach. First we see that from $N = \psi^6 \alpha$, we have identically

$$N = g^{1/2} \alpha \quad (108)$$

as a consequence of the method. Therefore, time slices t and $t + \delta t$ have a relation that is manifestly “harmonic:”

$$\bar{\partial}_0 N + N^2 K = N \bar{\partial}_0 \log \alpha , \quad (109)$$

a result that is fully consistent with our previous discussions and requiring that K be a fixed, non-scaling, variable.

Finally, we can establish the final relationships between the full riemannian metrics $g_{ij}(t)$ and $g'_{ij} = g_{ij}(t + \delta t)$ on the two manifestly harmonically related slices t and $t + \delta t$. As in (95),

$$\partial_t g_{ij} = \partial_t (\psi^4 \gamma_{ij}) = g_{ik} g_{jl} \left[-2N(A^{kl} + \frac{1}{3}g^{kl}K) + (\nabla^k \beta^l + \nabla^l \beta^k) \right] . \quad (110)$$

Working out (110) gives

$$\begin{aligned} \partial_t g_{ij} &= \psi^4 [u_{ij} + \gamma_{ij} \partial_t (\psi \log \psi)] \\ &= V_{ij} + g_{ij} \partial_t (\psi \log \psi) , \end{aligned} \quad (111)$$

where

$$\begin{aligned}\partial_t(\psi \log \psi) &= \frac{2}{3} (D_k \beta^k + 6\beta^k \partial_k \log \psi - \alpha K \psi^6) \\ &= \partial_t(g/\gamma)^{1/2} = \partial_t(g)^{1/3} = \frac{2}{3} (\nabla_k \beta^k - NK) .\end{aligned}\quad (112)$$

Hence, $\partial_t \psi$ and $\partial_t g_{ij}$ are fully determined and we note that the no-scaling rules for β^k and K were essential.

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